

# STABILITY OF AVERAGE ROUGHNESS, OCTAHEDRALITY, AND STRONG DIAMETER TWO PROPERTIES OF BANACH SPACES WITH RESPECT TO ABSOLUTE SUMS

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**ABSTRACT.** We prove that, if Banach spaces  $X$  and  $Y$  are  $\delta$ -average rough, then their direct sum with respect to an absolute norm  $N$  is  $\delta/N(1,1)$ -average rough. In particular, for octahedral  $X$  and  $Y$  and for  $p$  in  $(1, \infty)$  the space  $X \oplus_p Y$  is  $2^{1-1/p}$ -average rough, which is in general optimal. Another consequence is that for any  $\delta$  in  $(1, 2]$  there is a Banach space which is exactly  $\delta$ -average rough. We give a complete characterization when an absolute sum of two Banach spaces is octahedral or has the strong diameter 2 property. However, among all of the absolute sums, the diametral strong diameter 2 property is stable only for 1- and  $\infty$ -sums.

## 1. INTRODUCTION

A real Banach space  $X$  is said to be *octahedral* if, for every finite-dimensional subspace  $E$  of  $X$  and every  $\varepsilon > 0$ , there is a norm one element  $y \in X$  such that

$$\|x + y\| \geq (1 - \varepsilon)(\|x\| + \|y\|) \quad \text{for all } x \in E.$$

Octahedral Banach spaces were introduced by Godefroy and Maurey [10] (see also [9]) in order to characterize Banach spaces containing an isomorphic copy of  $\ell_1$ . This notion has recently been useful in studying the diameter 2 properties (see [4], [5], [11], and [12]). It is known that octahedrality is stable by taking  $\ell_1$ - or  $\ell_\infty$ -sums, and it is not stable by taking  $\ell_p$ -sums for  $p \in (1, \infty)$  (see [11, Proposition 3.12]). More precisely, for nontrivial Banach spaces  $X$  and  $Y$ ,

- if  $X$  or  $Y$  is octahedral, then  $X \oplus_1 Y$  is octahedral;
- if  $X$  and  $Y$  are both octahedral, then  $X \oplus_\infty Y$  is octahedral;
- $X \oplus_p Y$  is not octahedral for  $p \in (1, \infty)$ .

We extend these results quantitatively in two directions, instead of octahedral spaces we consider more general average rough spaces, and we also consider absolute normalized norm on direct sum.

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Let  $\delta > 0$ . A Banach space  $X$  is said to be  $\delta$ -average rough [8] if, whenever  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in S_X$ ,

$$\limsup_{\|y\| \rightarrow 0} \frac{1}{n} \sum_{i=1}^n \frac{\|x_i + y\| + \|x_i - y\| - 2}{\|y\|} \geq \delta.$$

Banach spaces which are 2-average rough are exactly the octahedral ones (see [4], [8], and [9]).

We recall that a norm  $N$  on  $\mathbb{R}^2$  is called *absolute* (see [7]) if

$$N(a, b) = N(|a|, |b|) \quad \text{for all } (a, b) \in \mathbb{R}^2$$

and *normalized* if

$$N(1, 0) = N(0, 1) = 1.$$

For example, the  $\ell_p$ -norm  $\|\cdot\|_p$  is absolute and normalized for every  $p \in [1, \infty]$ . If  $N$  is an absolute normalized norm on  $\mathbb{R}^2$  (see [7, Lemmata 21.1 and 21.2]), then

- $\|(a, b)\|_\infty \leq N(a, b) \leq \|(a, b)\|_1$  for all  $(a, b) \in \mathbb{R}^2$ ;
- if  $(a, b), (c, d) \in \mathbb{R}^2$  with  $|a| \leq |c|$  and  $|b| \leq |d|$ , then

$$N(a, b) \leq N(c, d);$$

- the dual norm  $N^*$  on  $\mathbb{R}^2$  defined by

$$N^*(c, d) = \max_{N(a, b) \leq 1} (|ac| + |bd|) \quad \text{for all } (c, d) \in \mathbb{R}^2$$

is also absolute and normalized. Note that  $(N^*)^* = N$ .

If  $X$  and  $Y$  are Banach spaces and  $N$  is an absolute normalized norm on  $\mathbb{R}^2$ , then we denote by  $X \oplus_N Y$  the product space  $X \times Y$  with respect to the norm

$$\|(x, y)\|_N = N(\|x\|, \|y\|) \quad \text{for all } x \in X \text{ and } y \in Y.$$

In the special case where  $N$  is the  $\ell_p$ -norm, we write  $X \oplus_p Y$ . Note that  $(X \oplus_N Y)^* = X^* \oplus_{N^*} Y^*$ .

By a *slice* of  $B_X$  we mean a set of the form

$$S(B_X, x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where  $x^* \in S_{X^*}$  and  $\alpha > 0$ . A *convex combination of slices* is a set of the form  $\sum_{i=1}^n \lambda_i S_i$ , where  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \geq 0$  with  $\sum_{i=1}^n \lambda_i = 1$ , and  $S_1, \dots, S_n$  are slices of  $B_X$ .

A dual characterization of  $\delta$ -average roughness is well known. The dual space  $X^*$  is  $\delta$ -average rough if and only if the diameter of every convex combination of slices of  $B_X$  is greater than or equal to  $\delta$  [8, Theorem 2]. In particular,  $X^*$  is octahedral if and only if the diameter of every convex combination of slices of  $B_X$  is 2 (see also [4], [9], and [11]). According to [1], the latter extreme property of a Banach space  $X$  is known as the *strong diameter 2 property*. An important class of Banach spaces with the strong diameter 2 property and which are octahedral are the Daugavet spaces (see [1] and [4]).

In [6], it is proved that the only absolute sums which preserve the Daugavet property are the  $\ell_1$ - and  $\ell_\infty$ -sum. Surprisingly, there are many absolute norms which preserve octahedrality and the strong diameter 2 property (see Section 3).

Recently, Becerra Guerrero, López-Pérez, and Rueda Zoca introduced a sharper version of the strong diameter 2 property (see [3]). A Banach space  $X$  has the *diametral strong diameter 2 property* if for every convex combination  $C$  of relatively weakly open subsets of  $B_X$ , for every  $x \in C$  and  $\varepsilon > 0$  there is a  $y \in C$  such that

$$\|x - y\| > 1 + \|x\| - \varepsilon.$$

By [3], Daugavet spaces have the diametral strong diameter 2 property and the diametral strong diameter 2 property implies the strong diameter 2 property. The Banach space  $c_0$  is an example of a space with the strong diameter 2 property and failing the diametral strong diameter 2 property. As far as the authors know it is an open question posed in [3] whether there is a Banach space with the diametral strong diameter 2 property and failing the Daugavet property. Our preliminary idea to attack this question was to check whether besides  $\ell_1$ - and  $\ell_\infty$ -norm (see [3] and [13]) there are more absolute norms which preserve the diametral strong diameter 2 property. However, there are none (see Section 3).

We now describe the contents of this paper. In Section 2, we prove (see Theorem 2.4) that for  $\delta$ -average rough Banach spaces  $X$  and  $Y$  their absolute sum  $X \oplus_N Y$  is  $\gamma\delta$ -average rough, where  $\gamma > 0$  is such that  $\gamma N(\cdot) \leq \|\cdot\|_\infty$ . In particular, we get that, for  $1 < p < \infty$ , the  $\ell_p$ -sum  $X \oplus_p Y$  of octahedral Banach spaces  $X$  and  $Y$  is  $2^{1-1/p}$ -average rough (see Corollary 2.6). Moreover, this number  $2^{1-1/p}$  is in general the largest possible one (see Proposition 2.7). As a consequence, we obtain that for any  $\delta \in (1, 2]$  there is a Banach space which is exactly  $\delta$ -average rough (see Theorem 2.8). We end this section by describing when the  $\delta$ -average roughness passes down from the absolute sum to one of the factors (see Proposition 2.11).

In Section 3, we characterize those absolute norms for which the direct sum of two octahedral Banach spaces is octahedral (see Theorem 3.2). As a consequence, we can characterize those absolute norms for which the direct sum of two separable Banach spaces with the almost Daugavet property has the almost Daugavet property (see Corollary 3.3). By duality, we can characterize the absolute norms which preserve the strong diameter 2 property (see Theorem 3.5). We end this section by proving that, similarly to the Daugavet property, among all of the absolute norms the diametral strong diameter 2 property is stable only for  $\ell_1$ - and  $\ell_\infty$ -sums (see Corollary 3.8).

## 2. AVERAGE ROUGHNESS OF ABSOLUTE SUMS

We begin by pointing out some equivalent but sometimes more convenient formulations of average roughness, which are easily derived from the definition.

**Proposition 2.1.** *Let  $X$  be a Banach space and  $\delta > 0$ . The following assertions are equivalent:*

- (i)  $X$  is  $\delta$ -average rough;
- (ii) whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in X$ , and  $\varepsilon > 0$  there is a  $y \in X$  such that  $\|y\| \leq \varepsilon$  and

$$\frac{1}{n} \sum_{i=1}^n \left( \|x_i + y\| + \|x_i - y\| \right) > (\delta - \varepsilon)\|y\| + \frac{2}{n} \sum_{i=1}^n \|x_i\|;$$

- (iii) whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ , and  $\varepsilon > 0$  there is a  $y \in X$  such that  $\|y\| \leq \varepsilon$  and

$$\frac{1}{n} \sum_{i=1}^n \left( \|x_i + y\| + \|x_i - y\| \right) > (\delta - \varepsilon)\|y\| + 2;$$

*Remark.* The equivalences in Proposition 2.1 remain true if one of the following holds

- (a) one replaces  $\frac{1}{n} \sum_{i=1}^n$  with  $\sum_{i=1}^n \lambda_i$ , where  $\lambda_i > 0$  and  $\sum_{i=1}^n \lambda_i = 1$ ;
- (b) one replaces  $\|y\| \leq \varepsilon$  with  $\|y\| = \varepsilon$ .

The  $\ell_1$ -sum of two Banach spaces inherits its  $\delta$ -average roughness from one of the factors.

**Proposition 2.2.** *Let  $X$  and  $Y$  be Banach spaces. If  $X$  or  $Y$  is  $\delta$ -average rough for some  $\delta > 0$ , then  $X \oplus_1 Y$  is also  $\delta$ -average rough.*

*Proof.* We consider only the case where  $X$  is  $\delta$ -average rough. The case where  $Y$  is  $\delta$ -average rough is similar. We will prove that  $Z = X \oplus_1 Y$  is  $\delta$ -average rough. Let  $z_1 = (x_1, y_1), \dots, z_n = (x_n, y_n) \in S_Z$  and  $\varepsilon > 0$ . By Proposition 2.1, it suffices to show that there exists  $z = (x, y) \in Z$  such that  $\|z\|_1 = \varepsilon$  and

$$\frac{1}{n} \sum_{i=1}^n \left( \|z_i + z\|_1 + \|z_i - z\|_1 \right) \geq (\delta - \varepsilon)\|z\|_1 + 2.$$

Since  $X$  is  $\delta$ -average rough, there is an  $x \in X$  such that  $\|x\| = \varepsilon$  and

$$\sum_{i=1}^n \frac{1}{n} \left( \|x_i + x\| + \|x_i - x\| \right) \geq (\delta - \varepsilon)\|x\| + \frac{2}{n} \sum_{i=1}^n \|x_i\|.$$

It follows that, for  $z = (x, 0)$  we have  $\|z\|_1 = \|x\| = \varepsilon$ , and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \left( \|z_i + z\|_1 + \|z_i - z\|_1 \right) \\
&= \frac{1}{n} \sum_{i=1}^n \left( \|x_i + x\| + \|y_i\| + \|x_i - x\| + \|y_i\| \right) \\
&\geq (\delta - \varepsilon)\|x\| + \frac{2}{n} \sum_{i=1}^n \|x_i\| + \frac{2}{n} \sum_{i=1}^n \|y_i\| \\
&= (\delta - \varepsilon)\|z\|_1 + 2.
\end{aligned}$$

□

**Corollary 2.3** (see [11, Proposition 3.12]). *If  $X$  or  $Y$  is octahedral, then  $X \oplus_1 Y$  is also octahedral.*

The following theorem is one of the main results in this section.

**Theorem 2.4.** *Let  $X$  and  $Y$  be Banach spaces,  $N$  an absolute normalized norm on  $\mathbb{R}^2$ , and  $\gamma > 0$  be such that  $\|\cdot\|_\infty \geq \gamma N(\cdot)$ . If  $X$  and  $Y$  are  $\delta$ -average rough for some  $\delta > 0$ , then  $X \oplus_N Y$  is  $\gamma\delta$ -average rough.*

*Proof.* Assume that  $X$  and  $Y$  are  $\delta$ -average rough. We will prove that  $Z = X \oplus_N Y$  is  $\gamma\delta$ -average rough. Let  $z_1 = (x_1, y_1), \dots, z_n = (x_n, y_n) \in S_Z$  and  $\varepsilon > 0$ . By Proposition 2.1, it suffices to show that there exists  $z = (x, y) \in Z$  such that  $\|z\|_N = \varepsilon N(1, 1)$  and

$$\frac{1}{n} \sum_{i=1}^n \left( \|z_i + z\|_N + \|z_i - z\|_N \right) \geq (\delta - \varepsilon)\gamma\|z\|_N + 2.$$

Choose  $c_i, d_i \geq 0$  such that  $N^*(c_i, d_i) = 1$  and  $c_i\|x_i\| + d_i\|y_i\| = 1$ . Denote by

$$c = \frac{1}{n} \sum_{i=1}^n c_i \quad \text{and} \quad d = \frac{1}{n} \sum_{i=1}^n d_i.$$

Note that  $c + d \geq 1$ , because  $c_i + d_i \geq N^*(c_i, d_i) = 1$ . Consider first the case where  $c \neq 0$  and  $d \neq 0$ . Denote by

$$\mu_i = \frac{1}{n} \frac{c_i}{c} \quad \text{and} \quad \nu_i = \frac{1}{n} \frac{d_i}{d}.$$

Observe that  $\mu_1 + \dots + \mu_n = \nu_1 + \dots + \nu_n = 1$ . Since  $X$  and  $Y$  are  $\delta$ -average rough, by Proposition 2.1, there are  $x \in X$  and  $y \in Y$  such that  $\|x\| = \|y\| = \varepsilon$  and

$$\sum_{i=1}^n \mu_i \left( \|x_i + x\| + \|x_i - x\| \right) \geq (\delta - \varepsilon)\|x\| + 2 \sum_{i=1}^n \mu_i \|x_i\|$$

and

$$\sum_{i=1}^n \nu_i \left( \|y_i + y\| + \|y_i - y\| \right) \geq (\delta - \varepsilon)\|y\| + 2 \sum_{i=1}^n \nu_i \|y_i\|.$$

It follows that, for  $z = (x, y)$  we have  $\|z\|_N = \varepsilon N(1, 1)$ , and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \left( \|z_i + z\|_N + \|z_i - z\|_N \right) \\
& \geq \frac{1}{n} \sum_{i=1}^n N(\|x_i + x\| + \|x_i - x\|, \|y_i + y\| + \|y_i - y\|) \\
& \geq \frac{1}{n} \sum_{i=1}^n \left( c_i(\|x_i + x\| + \|x_i - x\|) + d_i(\|y_i + y\| + \|y_i - y\|) \right) \\
& = c \sum_{i=1}^n \mu_i(\|x_i + x\| + \|x_i - x\|) + d \sum_{i=1}^n \nu_i(\|y_i + y\| + \|y_i - y\|) \\
& \geq c \left( (\delta - \varepsilon)\|x\| + 2 \sum_{i=1}^n \mu_i\|x_i\| \right) + d \left( (\delta - \varepsilon)\|y\| + 2 \sum_{i=1}^n \nu_i\|y_i\| \right) \\
& = (\delta - \varepsilon)(c\|x\| + d\|y\|) + \frac{2}{n} \sum_{i=1}^n (c_i\|x_i\| + d_i\|y_i\|) \\
& = (\delta - \varepsilon)(c + d) \max\{\|x\|, \|y\|\} + 2 \\
& \geq (\delta - \varepsilon)\gamma N(\|x\|, \|y\|) + 2 \\
& = (\delta - \varepsilon)\gamma\|z\|_N + 2.
\end{aligned}$$

Consider now the case where  $c = 0$ , which means that  $c_i = 0$  and  $d_i = 1$  for all  $i \in \{1, \dots, n\}$ . This implies that  $\|y_i\| = 1$  for all  $i \in \{1, \dots, n\}$ . Since  $Y$  is  $\delta$ -average rough, by Proposition 2.1, there exists a  $y \in Y$  such that  $\|y\| = \varepsilon N(1, 1)$  and

$$\sum_{i=1}^n \frac{1}{n} \left( \|y_i + y\| + \|y_i - y\| \right) \geq (\delta - \varepsilon)\|y\| + 2.$$

Therefore, for  $z = (0, y)$  we have  $\|z\|_N = \|y\| = \varepsilon N(1, 1)$ , and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \left( \|z_i + z\|_N + \|z_i - z\|_N \right) \\
& \geq \frac{1}{n} \sum_{i=1}^n \left( \|y_i + y\| + \|y_i - y\| \right) \\
& \geq (\delta - \varepsilon)\|y\| + 2 \\
& \geq (\delta - \varepsilon)\gamma\|z\|_N + 2.
\end{aligned}$$

The case where  $d = 0$  is similar to the case  $c = 0$ . We have thus proved that  $X \oplus_N Y$  is  $\gamma\delta$ -average rough.  $\square$

In particular, Theorem 2.4 applies to  $\ell_p$ -norms.

**Corollary 2.5.** *If Banach spaces  $X$  and  $Y$  are  $\delta$ -average rough for some  $\delta > 0$ , then*

- (a)  $X \oplus_\infty Y$  is  $\delta$ -average rough;
- (b)  $X \oplus_p Y$  is  $2^{-1/p}$ -average rough for  $1 < p < \infty$ .

**Corollary 2.6.** *If Banach spaces  $X$  and  $Y$  are octahedral and  $1 < p < \infty$ , then  $X \oplus_p Y$  is  $2^{1-1/p}$ -average rough.*

In Corollary 2.6, we saw that if  $X$  and  $Y$  are octahedral and  $1 < p < \infty$ , then  $X \oplus_p Y$  is  $2^{1-1/p}$ -average rough. We will now prove that in general  $2^{1-1/p}$  is the largest possible number.

**Proposition 2.7.** *Let  $X$  and  $Y$  be Banach spaces and  $1 < p < \infty$ . Then  $X \oplus_p Y$  is not  $\delta$ -average rough for any  $\delta > 2^{1-1/p}$ .*

*Proof.* We will prove that  $Z = X \oplus_p Y$  is not  $\delta$ -average rough for any  $\delta > 2^{1-1/p}$ . Consider the elements  $z_1 = (x_0, 0)$  and  $z_2 = (0, y_0)$  in  $Z$ , where  $x_0 \in S_X$  and  $y_0 \in S_Y$ . It suffices to show that there is a function  $f : (0, \infty) \rightarrow \mathbb{R}$  such that  $f(\varepsilon) \rightarrow 0$ , when  $\varepsilon \rightarrow 0$ , and that for every  $\varepsilon > 0$  and  $z \in Z$ , where  $\|z\| = \varepsilon$ ,

$$\frac{1}{2} \left( \|z_1 + z\|_p + \|z_1 - z\|_p + \|z_2 + z\|_p + \|z_2 - z\|_p \right) \leq \left( 2^{1-1/p} + f(\varepsilon) \right) \|z\|_p + 2.$$

Let  $\varepsilon \in (0, 1)$ . Let  $z = (x, y) \in Z$  be such that  $\|z\|_p = \varepsilon$ . By Maclaurin's formula,

$$(1 + \|x\|)^p = 1 + p\|x\| + \frac{p(p-1)}{2}(1 + \xi)^{p-2}\|x\|^2,$$

for some  $\xi \in (0, \|x\|)$ . Observe that

$$\begin{aligned} \|z_1 \pm z\|_p^p &= \|x_0 \pm x\|^p + \|y\|^p \leq (1 + \|x\|)^p + \|y\|^p \\ (2.1) \quad &= 1 + p\|x\| + \frac{p(p-1)(1 + \xi)^{p-2}}{2}\|x\|^2 + \|y\|^p. \end{aligned}$$

We continue by considering the cases  $1 < p \leq 2$  and  $p > 2$  separately. In both cases we will use the generalized Bernoulli's inequality, which says that for any  $t \geq 0$  we have  $(1 + t)^{1/p} \leq 1 + t/p$ .

**Case I.** Assume that  $1 < p \leq 2$ . Since  $\xi \in (0, \|x\|)$ , we have

$$(1 + \xi)^{p-2} \leq (1 + 0)^{p-2} = 1.$$

Combining the estimate (2.1) with Bernoulli's inequality we get

$$\begin{aligned} \|z_1 \pm z\|_p &\leq \left( 1 + p\|x\| + \frac{p(p-1)}{2}\|x\|^2 + \|y\|^p \right)^{1/p} \\ &\leq 1 + \|x\| + \frac{p-1}{2}\|x\|^2 + \frac{\|y\|^p}{p}. \end{aligned}$$

Similarly, we obtain

$$\|z_2 \pm z\|_p \leq 1 + \frac{\|x\|^p}{p} + \frac{p-1}{2}\|y\|^2 + \|y\|.$$

Therefore

$$\begin{aligned}
& \frac{1}{2} \left( \|z_1 + z\|_p + \|z_1 - z\|_p + \|z_2 + z\|_p + \|z_2 - z\|_p \right) \\
& \leq \left( 1 + \|x\| + \frac{p-1}{2} \|x\|^2 + \frac{\|y\|^p}{p} \right) + \left( 1 + \frac{\|x\|^p}{p} + \frac{p-1}{2} \|y\|^2 + \|y\| \right) \\
& = 2 + \|x\| + \|y\| + \frac{p-1}{2} (\|x\|^2 + \|y\|^2) + \frac{1}{p} (\|x\|^p + \|y\|^p) \\
& \leq 2 + 2^{1-1/p} \|(x, y)\|_p + \frac{p-1}{2} \varepsilon^2 + \frac{\varepsilon^p}{p} \\
& = 2 + \left( 2^{1-1/p} + \frac{p-1}{2} \varepsilon + \frac{\varepsilon^{p-1}}{p} \right) \|z\|_p.
\end{aligned}$$

Thus, for  $1 < p \leq 2$ , we can take

$$f(\varepsilon) = \frac{p-1}{2} \varepsilon + \frac{\varepsilon^{p-1}}{p}.$$

**Case II.** Assume that  $p > 2$ . Since  $\xi \in (0, \|x\|)$  and  $\|x\| \leq \varepsilon < 1$ , we have

$$(1 + \xi)^{p-2} \leq (1 + \|x\|)^{p-2} \leq (1 + \varepsilon)^{p-2} < 2^{p-2}.$$

Combining this estimate with (2.1) and Bernoulli's inequality, we get

$$\begin{aligned}
\|z_1 \pm z\|_p & \leq (1 + p\|x\| + p(p-1)2^{p-3}\|x\|^2 + \|y\|^p)^{1/p} \\
& \leq 1 + \|x\| + (p-1)2^{p-3}\|x\|^2 + \frac{\|y\|^p}{p}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\|z_2 \pm z\|_p & \leq (\|x\|^p + 1 + p\|y\| + p(p-1)2^{p-3}\varepsilon^2)^{1/p} \\
& \leq 1 + \frac{\|x\|^p}{p} + (p-1)2^{p-3}\|y\|^2 + \|y\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{1}{2} \left( \|z_1 + z\|_p + \|z_1 - z\|_p + \|z_2 + z\|_p + \|z_2 - z\|_p \right) \\
& \leq \left( 1 + \|x\| + (p-1)2^{p-3}\varepsilon^2 + \frac{\|y\|^p}{p} \right) \\
& \quad + \left( 1 + \frac{\|x\|^p}{p} + (p-1)2^{p-3}\varepsilon^2 + \|y\| \right) \\
& = 2 + (\|x\| + \|y\|) + (p-1)2^{p-3}(\|x\|^2 + \|y\|^2) + \frac{1}{p}(\|x\|^p + \|y\|^p) \\
& \leq 2 + 2^{1-1/p} \|(x, y)\|_p + (p-1)2^{p-3}2^{1-2/p}\varepsilon^2 + \frac{\varepsilon^p}{p} \\
& = \left( 2^{1-1/p} + (p-1)2^{p-2/p-2}\varepsilon + \frac{\varepsilon^{p-1}}{p} \right) \|z\| + 2.
\end{aligned}$$



Thus, for  $p > 2$ , we can take

$$f(\varepsilon) = (p-1)2^{p-2/p-2}\varepsilon + \frac{\varepsilon^{p-1}}{p}.$$

Hence  $X \oplus_p Y$  is not  $\delta$ -average rough for any  $\delta > 2^{1-1/p}$ .  $\square$

Now we are ready to show that for any  $\delta \in (1, 2]$  there is a Banach space which is exactly  $\delta$ -average rough.

**Theorem 2.8.** *For any  $\delta \in (1, 2]$  there is a dual Banach space, which is  $\delta$ -average rough and is not  $\gamma$ -average rough for any  $\gamma > \delta$ .*

*Proof.* If  $\delta = 2$ , then we can take  $\ell_1$ . If  $\delta \in (1, 2)$ , then there is a  $q \in (1, \infty)$  such that  $\delta = 2^{1/q}$ . Let  $p \in (1, \infty)$  be such that  $1/p + 1/q = 1$ . Since  $\ell_1$  is octahedral, then by Corollary 2.6 and Proposition 2.7 the Banach space  $\ell_1 \oplus_p \ell_1$  is  $\delta$ -average rough and is not  $\gamma$ -average rough for any  $\gamma > \delta$ .  $\square$

*Remark.* We do not know whether a similar result to Theorem 2.8 holds for  $\delta \in (0, 1]$ .

Theorem 2.8 and the dual characterization of  $\delta$ -average rough norms (see [8, Theorem 2]) immediately implies the following.

**Corollary 2.9.** *For any  $\delta \in (1, 2]$  there is a Banach space in which the minimal diameter of convex combination of slices is  $\delta$ .*

We end this section by describing when the  $\delta$ -average roughness passes down from the absolute sum to one of the factors. Our results are inspired by [2, Proposition 2.5].

The following lemma is easily verified from the definitions.

**Lemma 2.10.** *Let  $N$  be an absolute normalized norm on  $\mathbb{R}^2$  such that  $(1, 0)$  is an extreme point of the unit ball  $B_{(\mathbb{R}^2, N)}$ . Then  $(1, 0)$  is a strongly exposed point of  $B_{(\mathbb{R}^2, N)}$ , which is strongly exposed by the functional  $(1, 0) \in B_{(\mathbb{R}^2, N^*)}$ . In particular, for every  $\varepsilon > 0$  there is a  $\gamma > 0$  such that, whenever  $(a, b) \in B_{(\mathbb{R}^2, N)}$  and  $a > 1 - \gamma$ , then  $|b| < \varepsilon$ .*

**Proposition 2.11.** *Let  $X$  and  $Y$  be Banach spaces and  $N$  an absolute normalized norm on  $\mathbb{R}^2$  such that  $(1, 0)$  is an extreme point of  $B_{(\mathbb{R}^2, N^*)}$ . If  $X \oplus_N Y$  is  $\delta$ -average rough for some  $\delta > 0$ , then  $X$  is  $\delta$ -average rough.*

*Proof.* Assume that  $Z = X \oplus_N Y$  is  $\delta$ -average rough. Let  $x_1, \dots, x_n \in S_X$  and  $\varepsilon \in (0, \delta)$ . We will show that there is a  $u \in X$  such that  $\|u\| \leq \varepsilon$  and

$$\frac{1}{n} \sum_{i=1}^n (\|x_i + u\| + \|x_i - u\|) > (\delta - \varepsilon)\|u\| + 2.$$

By Lemma 2.10, there is a  $\gamma \in (0, \frac{2\varepsilon}{3})$  such that, whenever  $N^*(a, b) \leq 1$  and  $a > 1 - \gamma$ , then  $|b| < \frac{\varepsilon}{3}$ .

Consider  $(x_i, 0) \in S_Z$ . Since  $Z$  is  $\delta$ -average rough, there is a  $z = (u, v) \in Z$  such that  $\|z\|_N = \frac{\gamma}{2n}$  and

$$\frac{1}{n} \sum_{i=1}^n \left( \|(x_i, 0) + (u, v)\|_N + \|(x_i, 0) - (u, v)\|_N \right) > (\delta - \gamma/2)\|z\|_N + 2.$$

Choose  $a_i, b_i, c_i, d_i \geq 0$  with  $N^*(a_i, b_i) = N^*(c_i, d_i) = 1$  such that

$$a_i\|x_i + u\| + b_i\|v\| = N(\|x_i + u\|, \|v\|)$$

and

$$c_i\|x_i - u\| + d_i\|v\| = N(\|x_i - u\|, \|v\|).$$

Then we have

$$\frac{1}{n} \sum_{i=1}^n \left( a_i(\|x_i\| + \|u\|) + b_i\|v\| + c_i(\|x_i\| + \|u\|) + d_i\|v\| \right) > (\delta - \gamma/2)\|z\|_N + 2,$$

which implies that

$$\frac{a_i}{n} + \frac{n-1}{n} + 1 + 2\|z\|_N > (\delta - \gamma/2)\|z\|_N + 2.$$

It follows that  $a_i > 1 - \gamma$  and hence  $b_i < \varepsilon/3$  for all  $i \in \{1, \dots, n\}$ . Similarly, one obtains that  $c_i > 1 - \gamma$  and  $d_i < \varepsilon/3$  for all  $i \in \{1, \dots, n\}$ .

Therefore

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \left( \|x_i + u\| + \|x_i - u\| \right) \\ & \geq \frac{1}{n} \sum_{i=1}^n \left( a_i\|x_i + u\| \pm b_i\|v\| + c_i\|x_i - u\| \pm d_i\|v\| \right) \\ & > (\delta - \gamma/2)\|z\|_N + 2 - 2\frac{\varepsilon}{3}\|z\|_N \\ & = (\delta - \gamma/2 - 2\frac{\varepsilon}{3})\|z\|_N + 2 \\ & > (\delta - \varepsilon)\|u\| + 2. \end{aligned}$$

□

*Remark.* One can prove similarly to Proposition 2.11 that, if  $N$  is an absolute normalized norm on  $\mathbb{R}^2$  such that  $(0, 1)$  is an extreme point of  $B_{(\mathbb{R}^2, N^*)}$  and  $X \oplus_N Y$  is  $\delta$ -average rough for some  $\delta > 0$ , then  $Y$  is  $\delta$ -average rough.

**Corollary 2.12.** *If  $X \oplus_p Y$  is  $\delta$ -average rough and  $1 < p \leq \infty$ , then  $X$  and  $Y$  are  $\delta$ -average rough.*

### 3. OCTAHEDRALITY AND STRONG DIAMETER TWO PROPERTIES OF ABSOLUTE SUMS

In this section, we characterize those absolute norms for which the direct sum of two octahedral Banach spaces is octahedral. In fact, there are many such norms besides the  $\ell_1$ - and  $\ell_\infty$ -norm. Since octahedrality and the strong diameter 2 property are dually connected, it follows that there are many absolute norms which preserve the strong diameter 2 property. In order to present these characterizations we will introduce the notions of positive octahedrality and the positive strong diameter 2 property. We end this section by proving that, similarly to the Daugavet property, among all of the absolute norms the diametral strong diameter 2 property is stable only for  $\ell_1$ - and  $\ell_\infty$ -sums.

We begin by recalling the following equivalent formulation of octahedrality from [11].

**Proposition 3.1** (see [11, Proposition 2.2]). *Let  $X$  be a Banach space. The following assertions are equivalent:*

- (i)  $X$  is octahedral;
- (ii) whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ , and  $\varepsilon > 0$ , there is a  $y \in S_X$  such that

$$\|x_i + y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

**Definition.** An element  $(a, b) \in \mathbb{R}^2$  is *positive* if  $a \geq 0$  and  $b \geq 0$ . Let  $N$  be an absolute normalized norm on  $\mathbb{R}^2$ . We say that  $(\mathbb{R}^2, N)$  is *positively octahedral* if whenever  $n \in \mathbb{N}$  and positive  $(a_1, b_1), \dots, (a_n, b_n) \in S_{(\mathbb{R}^2, N)}$  there is a positive  $(c, d) \in S_{(\mathbb{R}^2, N)}$  such that

$$N((a_i, b_i) + (c, d)) = 2 \quad \text{for all } i \in \{1, \dots, n\}.$$

*Remark.* Note that  $(\mathbb{R}^2, N)$  is positively octahedral if and only if there is a  $(c, d) \in S_{(\mathbb{R}^2, N)}$  such that

$$N((1, 0) + (c, d)) = 2 \quad \text{and} \quad N((0, 1) + (c, d)) = 2.$$

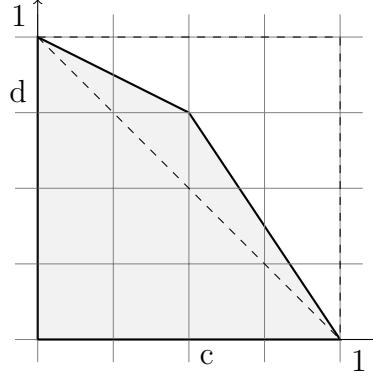


FIGURE 1. First quadrant of the unit ball of a positively octahedral  $(\mathbb{R}^2, N)$ .

**Theorem 3.2.** *Let  $X$  and  $Y$  be octahedral Banach spaces and  $N$  an absolute normalized norm on  $\mathbb{R}^2$ . Then  $X \oplus_N Y$  is octahedral if and only if  $(\mathbb{R}^2, N)$  is positively octahedral.*

*Proof. Necessity.* Assume that  $X \oplus_N Y$  is octahedral. Let  $\varepsilon > 0$  and positive  $(a_1, b_1), \dots, (a_n, b_n) \in S_{(\mathbb{R}^2, N)}$ . We will show that there is a positive  $(c, d) \in S_{(\mathbb{R}^2, N)}$  such that

$$N((a_i, b_i) + (c, d)) > 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Let  $x_i \in X$  and  $y_i \in Y$  be such that  $\|x_i\| = a_i$  and  $\|y_i\| = b_i$ . Since  $X \oplus_N Y$  is octahedral, there exists a  $(u, v) \in S_{X \oplus_N Y}$  such that  $\|(u, v)\|_N = 1$  and

$$\|(x_i, y_i) + (u, v)\|_N > 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Take  $c = \|u\|$  and  $d = \|v\|$ . Then for every  $i$

$$\begin{aligned} N((a_i, b_i) + (c, d)) &= N(a_i + c, b_i + d) \\ &= N(\|x_i\| + \|u\|, \|y_i\| + \|v\|) \\ &\geq N(\|x_i + u\|, \|y_i + v\|) \\ &> 2 - \varepsilon. \end{aligned}$$

*Sufficiency.* Assume that  $(\mathbb{R}^2, N)$  is positively octahedral. Let  $(x_1, y_1), \dots, (x_n, y_n) \in X \oplus_N Y$  be with norm one and  $\varepsilon > 0$ . We will show that there is a  $(u, v) \in X \oplus_N Y$  with norm one such that

$$\|(x_i, y_i) + (u, v)\|_N \geq (1 - \varepsilon)(2 - \varepsilon) \quad \text{for all } i \in \{1, \dots, n\}.$$

Since  $(\mathbb{R}^2, N)$  is positively octahedral, there is a positive  $(c, d) \in S_{(\mathbb{R}^2, N)}$  such that

$$N(\|x_i\| + c, \|y_i\| + d) \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Since  $X$  and  $Y$  are octahedral, there are  $x \in S_X$  and  $y \in S_Y$  such that

$$\|x_i + tx\| \geq (1 - \varepsilon)(\|x_i\| + t) \quad \text{for all } t \geq 0$$

and

$$\|y_i + ty\| \geq (1 - \varepsilon)(\|y_i\| + t) \quad \text{for all } t \geq 0.$$

Take  $u = cx$  and  $v = dy$ . It follows that  $\|(u, v)\|_N = 1$  and

$$\begin{aligned} \|(x_i, y_i) + (u, v)\|_N &= N(\|x_i + cx\|, \|y_i + dy\|) \\ &\geq (1 - \varepsilon)N(\|x_i\| + c, \|y_i\| + d) \\ &\geq (1 - \varepsilon)(2 - \varepsilon). \end{aligned}$$

□

Recall (see [14]) that a Banach space  $X$  has the *almost Daugavet property* if there is a 1-norming subspace  $Y$  of  $X^*$  such that

$$\|Id + T\| = 1 + \|T\|$$

holds true for every rank-one operator  $T: X \rightarrow X$  of the form  $T = y^* \otimes x$ , where  $x \in X$  and  $y^* \in Y$ . This definition is a generalization of the well-known Daugavet property, where  $Y = X^*$ . In [15, Propositions 2.1 and 2.2], it is shown that if  $X$  and  $Y$  are separable Banach spaces with the almost Daugavet property, then  $X \oplus_1 Y$  and  $X \oplus_\infty Y$  have the almost Daugavet property too. Since the almost Daugavet property and octahedrality coincide for separable Banach spaces (see [14, Theorem 1.1]), we immediately get from Theorem 3.2 the following stability result for almost Daugavet spaces.

**Corollary 3.3.** *Let  $X$  and  $Y$  be separable Banach spaces with the almost Daugavet property and  $N$  an absolute normalized norm on  $\mathbb{R}^2$ . Then  $X \oplus_N Y$  has the almost Daugavet property if and only if  $(\mathbb{R}^2, N)$  is positively octahedral.*

In order to characterize those absolute norms which preserve the strong diameter 2 property, we introduce the following notion.

**Definition.** Let  $N$  be an absolute normalized norm on  $\mathbb{R}^2$ . We say that  $\mathbb{R}^2$  has the *positive strong diameter 2 property* if whenever  $n \in \mathbb{N}$ , positive  $f_1, \dots, f_n \in S_{(\mathbb{R}^2, N^*)}$ ,  $\alpha_1, \dots, \alpha_n > 0$ , and  $\lambda_1, \dots, \lambda_n \geq 0$  with  $\sum_{i=1}^n \lambda_i = 1$  there are positive  $(a_i, b_i) \in S(B_{(\mathbb{R}^2, N)})$ ,  $f_i, \alpha_i$  such that

$$N\left(\sum_{i=1}^n \lambda_i(a_i, b_i)\right) = 1.$$

*Remark.* Note that  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property if and only if there are  $a, b \geq 0$  such that  $N(a, 1) = N(1, b) = 1$  and  $N\left(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\right) = 1$ .

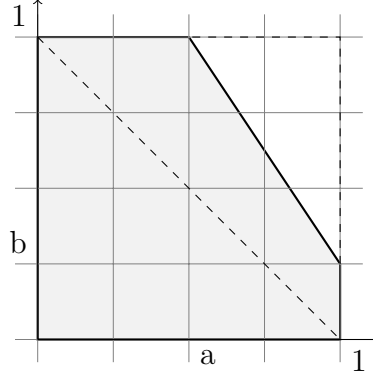


FIGURE 2. First quadrant of the unit ball of  $(\mathbb{R}^2, N)$  with the positive strong diameter 2 property.

**Proposition 3.4.** *Let  $N$  be an absolute normalized norm on  $\mathbb{R}^2$ . The space  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property if and only if  $(\mathbb{R}^2, N^*)$  is positively octahedral.*

*Proof. Necessity.* Assume that  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property. So there are  $a, b \geq 0$  such that  $N(a, 1) = N(1, b) = 1$  and

$$N\left(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\right) = 1.$$

Let  $c, d \geq 0$  be such that  $N^*(c, d) = 1$  and

$$(c, d)\left(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\right) = 1.$$

It implies that  $(c, d)(a, 1) = (c, d)(1, b) = 1$ . Hence

$$N^*((1, 0) + (c, d)) = ((1, 0) + (c, d))(1, b) = 2$$

and

$$N^*((0, 1) + (c, d)) = ((0, 1) + (c, d))(a, 1) = 2.$$

Therefore  $(\mathbb{R}^2, N^*)$  is positively octahedral.

*Sufficiency.* Assume now that  $(\mathbb{R}^2, N^*)$  is positively octahedral. So there exist  $c, d \geq 0$  such that  $N^*(c, d) = 1$  and

$$N^*((1, 0) + (c, d)) = 2 \quad \text{and} \quad N^*((0, 1) + (c, d)) = 2.$$

Let  $a, b, x, y \geq 0$  be such that  $N(a, y) = 1$ ,  $N(x, b) = 1$ ,

$$((1, 0) + (c, d))(x, b) = 2,$$

and

$$((0, 1) + (c, d))(a, y) = 2.$$

It follows that  $(1, 0)(x, b) = 1$  and  $(0, 1)(a, y) = 1$  which means that  $x = y = 1$ . Hence

$$N\left(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\right) = (c, d)\left(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property.  $\square$

The duality between the strong diameter 2 property and octahedrality, Theorem 3.2, and Proposition 3.4 yield the following result, however, we prefer to give its direct proof.

**Theorem 3.5.** *Let  $X$  and  $Y$  be Banach spaces with the strong diameter 2 property and  $N$  an absolute normalized norm on  $\mathbb{R}^2$ . Then  $X \oplus_N Y$  has the strong diameter 2 property if and only if  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property.*

*Proof. Necessity.* Assume that  $X \oplus_N Y$  has the strong diameter 2 property. We will show that  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property. Let  $(c_1, d_1), \dots, (c_n, d_n)$  be positive elements in  $S_{(\mathbb{R}^2, N^*)}$ ,  $\alpha_1, \dots, \alpha_n > 0$ ,  $\lambda_i > 0$  with  $\sum_{i=1}^n \lambda_i = 1$ , and  $\varepsilon > 0$ . We will show that there are positive  $(a_i, b_i) \in B_{(\mathbb{R}^2, N)}$  such that  $c_i a_i + d_i b_i > 1 - \alpha_i$  and  $N(\sum_{i=1}^n \lambda_i (a_i, b_i)) > 1 - \varepsilon$ .

Let  $(x_i^*, y_i^*) \in S_{X^* \oplus_N Y^*}$  be such that  $\|x_i^*\| = c_i$  and  $\|y_i^*\| = d_i$  for every  $i$ . Since  $X \oplus_N Y$  has the strong diameter 2 property, there are

$$(x_i, y_i) \in S(B_{X \oplus_N Y}, (x_i^*, y_i^*), \alpha_i)$$

such that  $\|\sum_{i=1}^n \lambda_i (x_i, y_i)\|_N \geq 1 - \varepsilon$ .

Take  $(a_i, b_i) = (\|x_i\|, \|y_i\|)$ . Then  $c_i a_i + d_i b_i > 1 - \alpha_i$ , because

$$c_i a_i + d_i b_i = \|x_i^*\| \|x_i\| + \|y_i^*\| \|y_i\| \geq x_i^*(x_i) + y_i^*(y_i) > 1 - \alpha_i$$

and

$$\begin{aligned} N\left(\sum_{i=1}^n \lambda_i (a_i, b_i)\right) &= N\left(\sum_{i=1}^n \lambda_i \|x_i\|, \sum_{i=1}^n \lambda_i \|y_i\|\right) \\ &\geq N\left(\left\|\sum_{i=1}^n \lambda_i x_i\right\|, \left\|\sum_{i=1}^n \lambda_i y_i\right\|\right) \\ &= \left\|\sum_{i=1}^n \lambda_i (x_i, y_i)\right\|_N \geq 1 - \varepsilon. \end{aligned}$$

*Sufficiency.* We use an idea from [13]. Assume that  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property. Let  $S_1, \dots, S_n$  be slices of  $B_{X \oplus_N Y}$  defined by norm one functionals  $(x_i^*, y_i^*)$  and scalars  $\alpha_i > 0$ . Let  $\lambda_i > 0$  be such that  $\sum_{i=1}^n \lambda_i = 1$ . We will show that the diameter of  $\sum_{i=1}^n \lambda_i S_i$  is 2.

Let  $\varepsilon > 0$ . Consider the slices  $S_i^X = S(B_X, \frac{x_i^*}{\|x_i^*\|}, \frac{\alpha_i}{2})$  and  $S_i^Y = S(B_Y, \frac{y_i^*}{\|y_i^*\|}, \frac{\alpha_i}{2})$  (If  $x_i^* = 0$ , then  $S_i^X = B_X$  and if  $y_i^* = 0$ , then  $S_i^Y = B_Y$ ).

Since  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property, there are positive  $(a_i, b_i) \in S(B_{(\mathbb{R}^2, N)}, (\|x_i^*\|, \|y_i^*\|), \delta)$  such that  $N\left(\sum_{i=1}^n \lambda_i (a_i, b_i)\right) > 1 - \delta$ , where  $\delta > 0$  satisfies  $(1 - \delta)(1 - \alpha_i/2) \geq 1 - \alpha_i$  for all  $i \in \{1, \dots, n\}$ .

It turns out that  $a_i S_i^X \times b_i S_i^Y \subset S_i$ . Indeed, if  $x \in S_i^X$  and  $y \in S_i^Y$ , then

$$\|(a_i x, b_i y)\|_N = N(a_i \|x\|, b_i \|y\|) \leq N(a_i, b_i) \leq 1$$

and

$$a_i x_i^*(x) + b_i y_i^*(y) > (1 - \delta)(1 - \frac{\alpha_i}{2}) \geq 1 - \alpha_i.$$

Denote by

$$a = \sum_{i=1}^n \lambda_i a_i \quad \text{and} \quad b = \sum_{i=1}^n \lambda_i b_i.$$

Suppose that  $a \neq 0$  and  $b \neq 0$ . For every  $i$ , denote by

$$\mu_i = \frac{\lambda_i a_i}{a} \quad \text{and} \quad \nu_i = \frac{\lambda_i b_i}{b}.$$

As  $X$  and  $Y$  have the strong diameter 2 property, then there are  $\hat{x}, \hat{u} \in \sum_{i=1}^n \mu_i S_i^X$  and  $\hat{y}, \hat{v} \in \sum_{i=1}^n \nu_i S_i^Y$  such that  $\|\hat{x} - \hat{u}\| \geq 2 - \varepsilon$  and  $\|\hat{y} - \hat{v}\| \geq 2 - \varepsilon$ . Take  $x = a\hat{x}$ ,  $y = b\hat{y}$ ,  $u = a\hat{u}$ , and  $v = b\hat{v}$ . Then  $(x, y), (u, v) \in \sum_{i=1}^n \lambda_i S_i$ , because  $x, u \in \sum_{i=1}^n \lambda_i a_i S_i^X$  and  $y, v \in \sum_{i=1}^n \lambda_i b_i S_i^Y$ . Finally,

$$\begin{aligned} \|(x, y) - (u, v)\|_N &= N(\|x - u\|, \|y - v\|) \\ &\geq (2 - \varepsilon)N(a, b) \\ &> (2 - \varepsilon)(1 - \delta). \end{aligned}$$

Consider now the case, where  $a = 0$  or  $b = 0$ . Assume that  $a = 0$ . Since

$$\{0\} \times S_i^Y \subset S_i,$$

then

$$\{0\} \times \sum_{i=1}^n \lambda_i S_i^Y \subset \sum_{i=1}^n \lambda_i S_i.$$

As the diameter of  $\sum_{i=1}^n \lambda_i S_i^Y$  is 2, there are  $y, v \in \sum_{i=1}^n \lambda_i S_i^Y$  such that

$$\|y - v\| \geq 2 - \varepsilon.$$

Thus  $(0, y), (0, v) \in \sum_{i=1}^n \lambda_i S_i$ . Now we have

$$\begin{aligned} \|(0, y) - (0, v)\|_N &= N(0, \|y - v\|) \\ &= \|y - v\| \\ &\geq 2 - \varepsilon. \end{aligned}$$

□

We now turn our attention to investigate the stability of the diametral strong diameter 2 property. From [3] and [13], we know that  $X \oplus_\infty Y$  and  $X \oplus_1 Y$  have the diametral strong diameter 2 property as soon as  $X$  and  $Y$  have the diametral strong diameter 2 property. We end this section by proving that there are no other absolute norms different from  $\ell_1$ - and  $\ell_\infty$ -norm which preserve the diametral strong diameter 2 property. Since the diametral strong diameter 2 property



implies the strong diameter 2 property and the latter is stable only for absolute norms with the positive strong diameter 2 property, we can restrict our attention to them.

Consider an absolute normalized norm  $N$  on  $\mathbb{R}^2$ , different from the  $\ell_1$ -norm and  $\ell_\infty$ -norm, such that  $(\mathbb{R}^2, N)$  has the positive strong diameter 2 property. Thus, for some  $a, b \in [0, 1)$  with  $a > 0$  or  $b > 0$ ,  $N$  is defined by

$$(3.1) \quad N(c, d) = \max \left\{ |c|, |d|, \frac{(1-b)|c| + (1-a)|d|}{1-ab} \right\} \quad \text{for all } (c, d) \in \mathbb{R}^2.$$

**Proposition 3.6.** *Let  $X$  and  $Y$  be nontrivial Banach spaces and  $N$  defined by (3.1). Then  $X \oplus_N Y$  does not have the diametral strong diameter 2 property.*

We will use the following elementary lemma.

**Lemma 3.7.** *There is a  $\lambda \in (0, 1)$  such that*

$$N(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b) < 1 + N(\lambda, 1-\lambda).$$

*Proof.* Assume that  $\lambda \in (0, 1)$ . Denote by

$$c = 2\lambda + (1-\lambda)a \quad \text{and} \quad d = 2(1-\lambda) + \lambda b.$$

It is straightforward to show directly that the condition

$$N(c, d) = \frac{(1-b)c + (1-a)d}{1-ab}$$

is equivalent to

$$\frac{a}{2+a-ab} \leq \lambda \leq \frac{2-ab}{2+b-ab},$$

and the condition

$$\frac{(1-b)c + (1-a)d}{1-ab} < 1 + N(\lambda, 1-\lambda)$$

is equivalent to

$$\lambda < \frac{a}{1+a} \quad \text{or} \quad \lambda > \frac{1}{1+b}.$$

Note that

$$\frac{a}{2+a-ab} \leq \frac{a}{1+a} \leq \frac{1}{1+b} \leq \frac{2-ab}{2+b-ab},$$

where the first inequality is strict if and only if  $a \neq 0$ , and the last inequality is strict if and only if  $b \neq 0$ .  $\square$

*Proof of Proposition 3.6.* By using Lemma 3.7, we choose  $\lambda \in (0, 1)$  such that

$$N(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b) < 1 + N(\lambda, (1-\lambda)).$$

Denote by

$$\delta = 1 + N(\lambda, 1 - \lambda) - N\left(2\lambda + (1 - \lambda)a, 2(1 - \lambda) + \lambda b\right).$$

Choose any  $\varepsilon \in (0, \delta/2)$ . Let  $\alpha > 0$  be such that if  $(c_1, d_1), (c_2, d_2) \in \mathbb{R}^2$  satisfy the conditions  $N(c_1, d_1), N(c_2, d_2) \leq 1$ ,  $|c_1| > 1 - \alpha$ , and  $|d_2| > 1 - \alpha$ , then

$$N\left(2\lambda + (1 - \lambda)|c_2|, 2(1 - \lambda) + \lambda|d_1|\right) \leq N\left(2\lambda + (1 - \lambda)a, 2(1 - \lambda) + \lambda b\right) + \varepsilon.$$

Fix any  $x^* \in S_{X^*}$  and  $y^* \in S_{Y^*}$ . Consider the slices  $S_1 = S(B_{X \oplus_N Y}, (x^*, 0), \alpha)$  and  $S_2 = S(B_{X \oplus_N Y}, (0, y^*), \alpha)$ . Choose  $x \in S_X$  and  $y \in S_Y$  such that  $(x, 0) \in S_1$  and  $(0, y) \in S_2$ . Assuming that the Banach space  $X \oplus_N Y$  has the diametral strong diameter 2 property, there exist  $(u_1, v_1) \in S_1$  and  $(u_2, v_2) \in S_2$  such that

$$\begin{aligned} \tilde{N} &:= \|\lambda(x, 0) + (1 - \lambda)(0, y) - \lambda(u_1, v_1) - (1 - \lambda)(u_2, v_2)\|_N \geq \\ &\geq \|\lambda(x, 0) + (1 - \lambda)(0, y)\|_N + 1 - \varepsilon. \end{aligned}$$

Since

$$\begin{aligned} \tilde{N} &= N\left(\|\lambda x - \lambda u_1 - (1 - \lambda)u_2\|, \|(1 - \lambda)y - \lambda v_1 - (1 - \lambda)v_2\|\right) \leq \\ &\leq N\left(2\lambda + (1 - \lambda)\|u_2\|, 2(1 - \lambda) + \lambda\|v_1\|\right) \leq \\ &\leq N\left(2\lambda + (1 - \lambda)a, 2(1 - \lambda) + \lambda b\right) + \varepsilon = \\ &= 1 + N(\lambda, 1 - \lambda) - \delta + \varepsilon, \end{aligned}$$

it follows that

$$\|\lambda(x, 0) + (1 - \lambda)(0, y)\|_N + 1 - \varepsilon \leq 1 + N(\lambda, 1 - \lambda) - \delta + \varepsilon,$$

i.e.,  $\delta \leq 2\varepsilon$ , which is a contradiction.  $\square$

Combining [3, Theorem 3.8], [13, Theorem], and Proposition 3.6, we get the following corollary.

**Corollary 3.8.** *If  $Z = X \oplus_N Y$  has the diametral strong diameter 2 property, then either  $Z = X \oplus_1 Y$  or  $Z = X \oplus_\infty Y$ .*

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